Recall:

$(l^p)^* = l^q$, for $p>1$. $(\frac{1}{p}+\frac{1}{q}=1)$

$(L^p)^* = L^q$, for $p>1$. $(\frac{1}{p}+\frac{1}{q}=1)$

$L^p = \{ f$ is Leb measurable fct : $\int |f|^p d\mathcal{L} < \infty \}$

For any $f \in (L^p)^*$, there exists a fct $y \in L^q$, st

$$f(x) = \int x(t)\, y(t)\, dt, \quad \forall x \in L^p$$

We denote $f = y^*$ and write $\langle y^*, x \rangle = y^*(x)$

Examples of adjoint operators on Banach space.

Eg 1: Let $X = Y = L^2[0,1]$ and define

$$Ax = \int_0^1 k(t,s)\, x(s)\, ds, \quad t \in [0,1]$$

where $\iint_0^1 |k(t,s)|^2 ds\, dt < \infty$

Compute A^*.

- Check $A \in B(X, Y)$

$$\int_0^1 |Ax|^2 dt = \int_0^1 \left| \int_0^1 k(t,s)\, x(s)\, ds \right|^2 dt$$

$\left(\|fg\|_1 \le \|f\|_p \|g\|_q \quad \frac{1}{p}+\frac{1}{q}=1 \quad \text{Here } p=q=2 \right) \rightarrow$

$$\le \int_0^1 \left[\left(\int_0^1 |k(t,s)|^2 ds \right)^{\frac{1}{2}} \cdot \left(\int_0^1 |x(s)|^2 ds \right)^{\frac{1}{2}} \right]^2 dt$$

$$= \int_0^1 \int_0^1 |k(t,s)|^2 ds \cdot \|x\|_2^2 \, dt$$

$$= \|x\|_2^2 \cdot \iint_0^1 |k(t,s)|^2 ds\, dt$$

$$< \infty$$

$Ax \in Y$.

Compute $A^* y^*$ (Since $A^* \in B(Y^*, X^*) = B(L^2, L^2)$).

$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$

$$\langle y^*, Ax \rangle = \int_0^1 y(t)\,(A(x)(t))\,dt$$
$$= \int_0^1 y(t)\left(\int_0^1 k(t,s)\, x(s)\, ds\right) dt$$
$$= \int_0^1 x(s)\left(\int_0^1 k(t,s)\, y(t)\, dt\right) ds$$

By interchanging the roles of s and t,

$$\langle y^*, Ax \rangle = \int_0^1 x(t) \left(\int_0^1 k(s,t)\, y(s)\, ds\right) dt$$

$||$

$\langle A^* y^*, x \rangle$

$$A^* y = A^* y^* = \int_0^1 k(s,t)\, y(s)\, ds$$

· The adjoint of A is obtained by interchanging s and t in K.

Eg 2 Again Let $X = Y = L_2[0,1]$ and define

$$Ax(t) = \int_0^t k(t,s)\, x(s)\, ds, \quad t \in [0,1]$$

with $\iint_0^1 |k(t,s)|^2\, dt\, ds < \infty$.

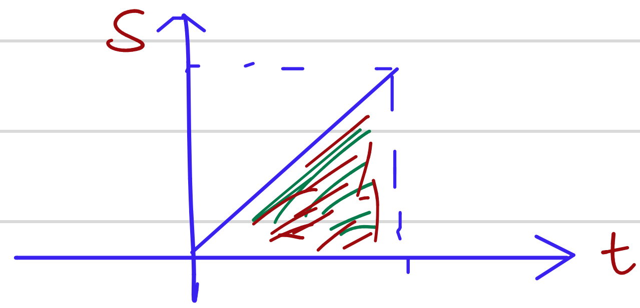
Compute A^*.

· Check $Ax \in L^2$. ✓

$$\langle y^*, Ax \rangle = \int_0^1 y(t) \int_0^t k(t,s)\, x(t)\, ds\, dt$$

$$\langle y^*, Ax\rangle = \int_0^1 y(t) \int_0^t k(t,s)\, x(t)\, ds\, dt$$

The double integration represents integration over the triangular region



Change the order of integration,

$$\langle y^*, Ax\rangle = \int_0^1 \int_s^1 y(t)\, k(t,s)\, x(s)\, dt\, ds$$

$$= \int_0^1 x(s) \left(\int_s^1 y(t)\, k(t,s)\, dt\right) ds$$

$$\langle A^* y^*, x\rangle = \int_0^1 x(t) \left(\int_t^1 y(s)\, k(s,t)\, ds\right) dt$$

$$\Rightarrow \quad A^* y = \int_t^1 y(s)\, k(s,t)\, ds$$

Remark: $A^* y \in L^2$ $\quad (\int |A^* y|^2 dt$

Eg3: Let $X = L_p[0,1]$, $1 < p < \infty$. $Y = L_q[0,1]$.
with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $T \in B(X, Y)$ be defined by

$$Tx = \int_0^1 k(t,s)\, x(s)\, ds$$

where $\int_0^1\int_0^1 |k(t,s)|^q\, dt\, ds < \infty$

Find A^*.

(for any $f \in L^{q*}$, $\exists y \in L^p$,

$f(x) = \int_0^1 x(t)\, y(t)\, dt$, $\quad \forall x \in L^q$,

$y^* := f$.

$$\langle T^* y^*, x \rangle = \langle y^*, Tx \rangle$$
$$= \int_0^1 x(s) \left(\int_0^1 k(t,s)\, y(t)\, dt \right) ds$$
$$= \int_0^1 x(t) \left(\int_0^1 k(s,t)\, y(s)\, ds \right) dt$$

$$T^* y = \int_0^1 k(s,t)\, y(s)\, ds$$

claim: $T^* y \in L_q$

$$\int_0^1 \left(\int_0^1 k(s,t)\, y(s)\, ds \right)^q dt$$

$$\leq \int_0^1 \left[\left(\int_0^1 |k(s,t)|^q ds \right)^{\frac{1}{q}} \cdot \left(\int_0^1 y(s)^p ds \right)^{\frac{1}{p}} \right]^q dt$$

$$\leq \iint_0^1 |k(s,t)|^q\, ds\, dt \cdot \| y \|_p^q$$

$$< \infty$$

4. Show that for any ~~sphere~~ $S(0,r)$ in a normed space X and point $x_0 \in S(0,r)$ there is a hyperplane H_0 through x_0 such that the Ball $\bar{B}(0,r)$ lies entirely in one of the two half planes determined by H_0.

Pf (Application of Hahn-Banach Thm.)

There exists $\tilde{f} \in X^*$ such that $\tilde{f}(x_0) = \|x_0\| = r$, $\|\tilde{f}\| = 1$.

Define $H_0 = \{x : \tilde{f}(x) = r\}$

Then we have $x_0 \in H_0$

Moremore, for any $x \in \bar{B}(0,r)$, it follows that

$$|\tilde{f}(x)| \le \|\tilde{f}\| \cdot \|x\| = \|x\| \le r$$

($A = \{x : \tilde{f}(x) < r\}$, $B = \{x : \tilde{f}(x) > r\}$.

Remark: Let X be a real normed space and let A, B be two non-empty disjoint convex subset of X.

(i) If A is open, then $\exists$ a bounded linear fct f on X and $c \in \mathbb{R}$, such that

$$f(a) < c \le f(b) \quad \text{for any } a \in A,\ b \in B.$$

(ii) If A is compuct and B is closed, then $\exists$ a bounded linear fct f on X and $c_1, c_2 \in \mathbb{R}$, st

$$f(a) \le c_1 < c_2 \le f(b), \quad \forall a \in A,\ b \in B$$